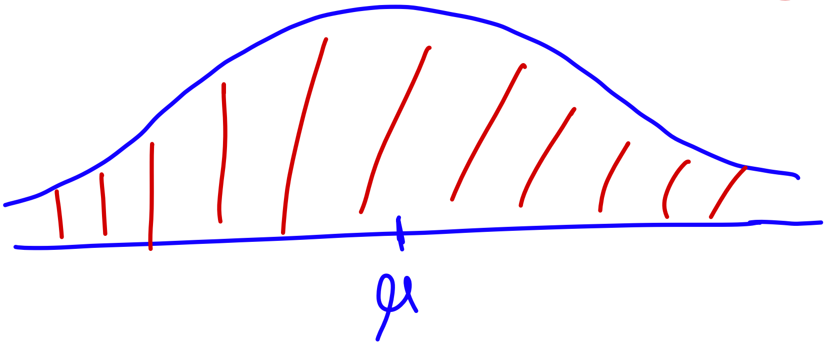
# Chapter 5 The normal distribution

التوزيع الطبيعي

→ Continous probability distribution



① Symmetric

② $\mu = \text{mode} = Q_2$

③ Total area $= 1$

مهمة

④ probability $=$ area

⑤ $P(X = k) = 0$

⑥ $P(X \le k) = P(X < k)$

ملاحظة المساواة ليس لها معنى في normal distribution

$$X \sim n(\mu, \sigma^2)$$

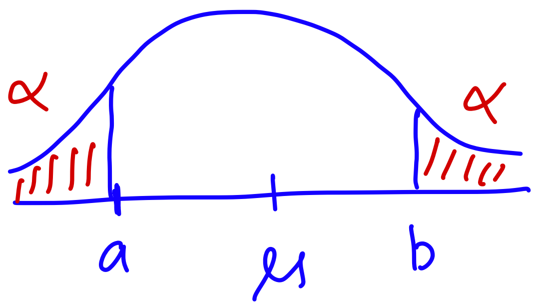
- follows
- is distributed as

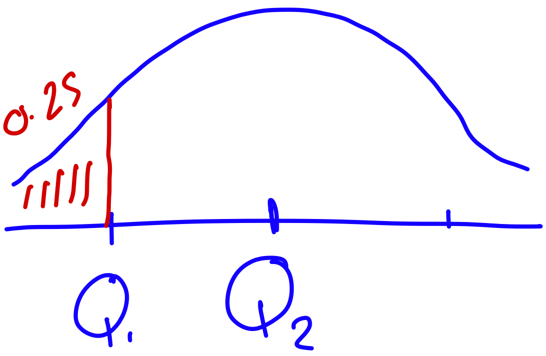
Examples of normal distribution:

- weights
- speed
- Blood pressure
- height

## NOTES

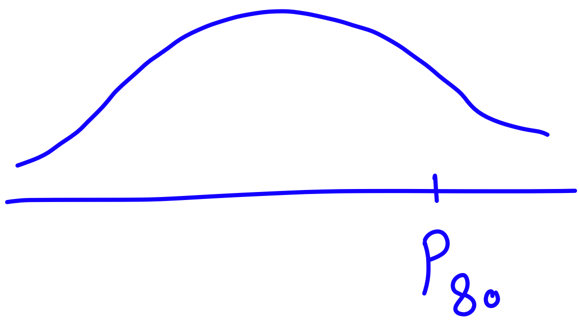
① $\mu = \dfrac{a+b}{2}$



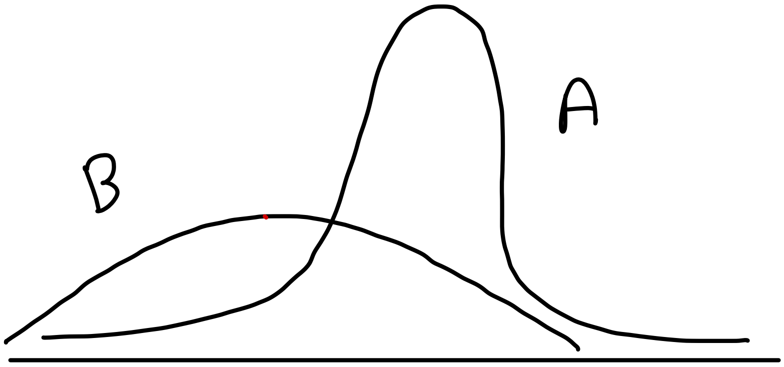

② 

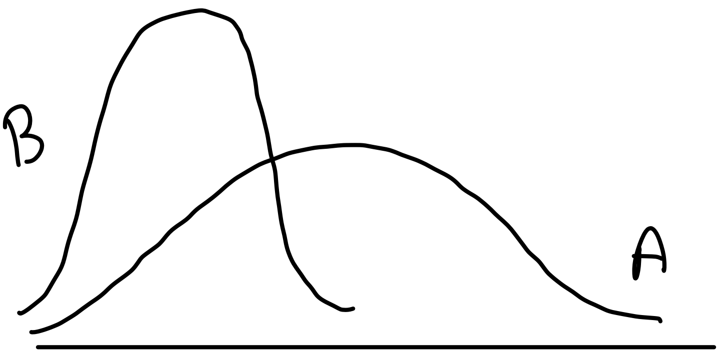

- $P(X \le Q_1) = 0.25$
- $P(X \le Q_3) = 0.75$

③ $P(X \le P_{80}) = 0.80$



**Example** A & B are normally distributed curves. which one has larger mean? which one has larger standard deviation?

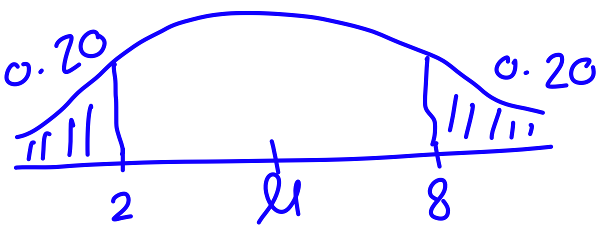



**Example** If $X \sim n(\mu, u)$, and

$P(X<2)=0.20$, $P(X>8)=0.20$, Find:

① $\mu = \dfrac{2+8}{2} = 5$



② $P(2 \le X < 8)$

$0.20 + X + 0.20 = 1$

$X + 0.40 = 1$ $\quad \boxed{X = 0.60}$

③ $20^{th}$ percentile $(P_{20})$

$P(X \le P_{20}) = 0.20$

$$\boxed{P_{20} = 2}$$

④ $80^{th}$ percentile $(P_{80})$

$P(X \le P_{80}) = 0.80$

$$\boxed{P_{80} = 8}$$

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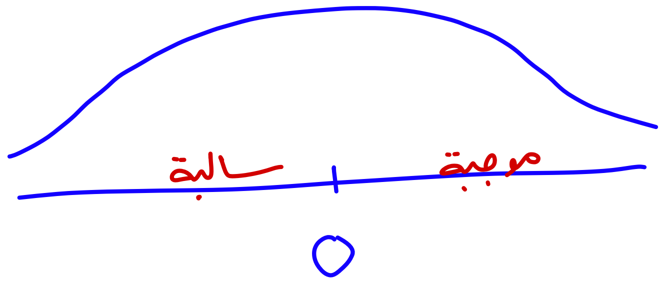
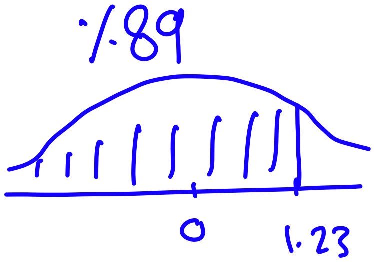
* The standard normal distribution

$$Z \sim n(0,1)$$




**Example** If $Z \sim n(0,1)$, then find:

i) $P(Z \leq 1.23) = 0.8907$



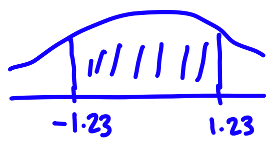

ii) $P(Z \geq 1.23) = 0.1093$

iii) $P(1.23 \leq Z \leq 2.12)$

$= P(Z \leq 2.12) - P(Z \leq 1.23)$

$= 0.9830 - 0.8907$

iv) $P(-1.23 < Z < 1.23)$




$= P(Z < 1.23) - P(Z < -1.23)$

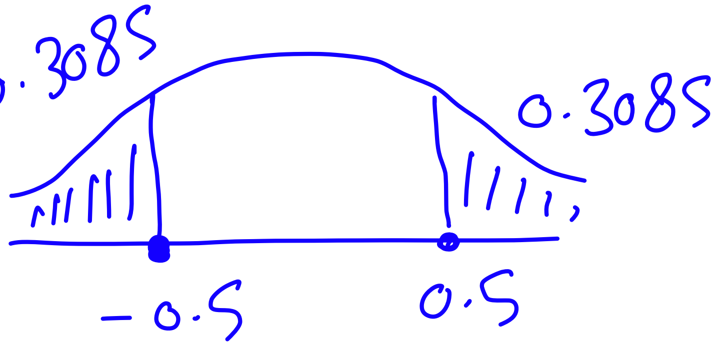
$= 0.8907 - 0.1093 = \_\_\_$

v) The z-score that correspond to a commulative area of 0.6915

$z = 0.5$

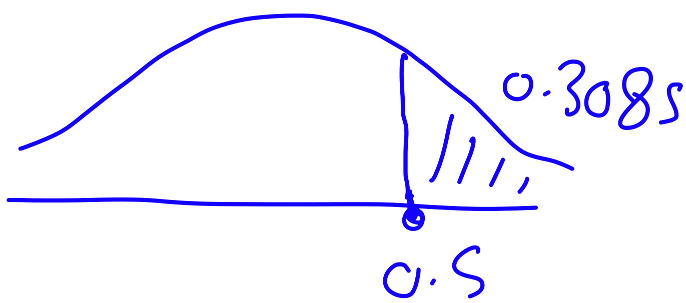

vi) If $p(z \leq a) = 0.3085$ find a?

$\boxed{a = -0.5}$



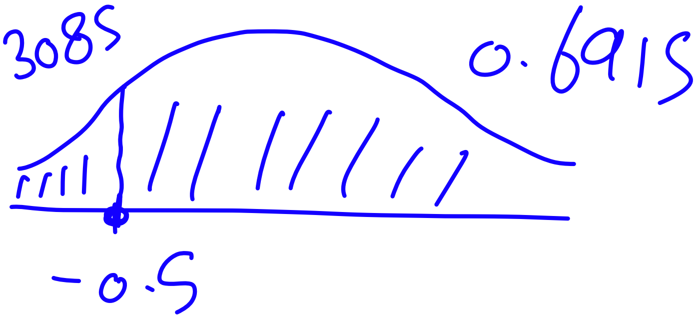

vii) a if $p(z > a) = 0.3085$ ?

$a = 0.5$



viii) $a$ if $P(Z \geq a) = 0.6915$

$\boxed{a = -0.5}$

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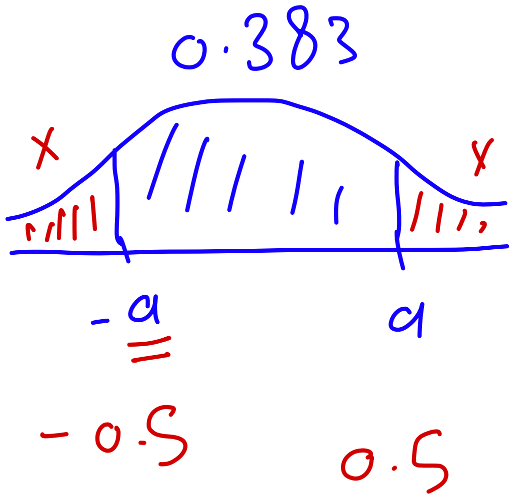
ix) $a$ if $P(-a < Z < a) = 0.383$

$P(Z < a) - P(Z < -a) = 0.383$

$x + x + 0.383 = 1$

$x = 0.3085$

$a = \pm 0.5$



## * Standardization

$$n \longrightarrow z$$

$$X \sim n(\mu, \sigma^2)$$

$$\Rightarrow Z = \frac{X-\mu}{\sigma} \sim n(0,1)$$

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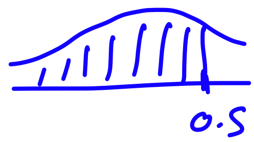
**Example** If $X \sim n(5, 4)$, find:

1) $P(X \le 6)$




$$= P\left(Z \le \frac{6-5}{2}\right) = P(Z \le 0.5)$$

$$= 0.6915$$

2) $P(X > 4) \Rightarrow P\left(Z > \frac{4-5}{2}\right)$

$$\Rightarrow P(Z > -0.5)$$

$$= 0.6915$$

3) $P(4 \le X \le 6)$

$= P(X \le 6) - P(X \le 4)$

$= P\left(Z \le \frac{6-5}{2}\right) - P\left(Z \le \frac{4-5}{2}\right)$

$= P(Z \le 0.5) - P(Z \le -0.5)$

$= 0.6915 - 0.3085$

4) $a$ if $P(X > a) = 0.6915$

$P\left(Z > \frac{a-5}{2}\right) = 0.6915$

$P(Z > -0.5) = 0.6915$

$-0.5 = \frac{a-5}{2}$ $\boxed{a = 4}$

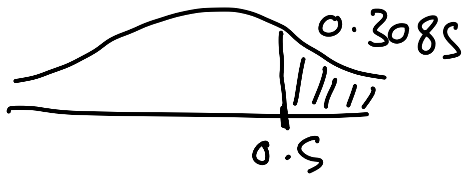
Example If $X \sim n(\mu, 4)$, $P(X \geq 6) = 0.3085$

find $\mu$?

$$\Rightarrow P(X \geq 6) = 0.3085$$

$$= P\left(Z \geq \frac{6-\mu}{2}\right) = 0.3085$$

$$\frac{1}{\cancel{2}} = \frac{6-\mu}{\cancel{2}}$$




$$1 = 6 - \mu \qquad \boxed{\mu = 5}$$

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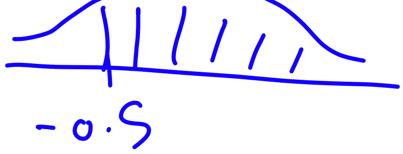
Example If $X \sim n(5, \sigma^2)$,

$P(X > 4) = 0.6915$ find $\sigma^2$?

$$\Rightarrow P(X > 4) = 0.6915$$

$$= P\left(Z > \frac{4-5}{\sigma}\right) = 0.6915$$

$$-\frac{1}{2} = \frac{-1}{\sigma}$$

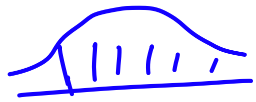


$\boxed{\sigma = 2}$ $\boxed{\sigma^2 = 4}$

**Example** If $X \sim N(\mu, \sigma^2)$, $P(X>4) = 0.6915$, $P(X \le 6) = 0.6915$, find $\mu, \sigma^2$?

$$\Rightarrow P(X > 4) = 0.6915$$

$$P\left(Z > \frac{4-\mu}{\sigma}\right) = 0.6915$$



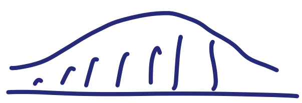
$$-\frac{1}{2} = \frac{(4-\mu)}{\sigma}$$

$$8 - 2\mu = -\sigma$$

$$\boxed{2\mu - \sigma = 8} \cdots\cdots (1)$$

$$\Rightarrow P(X \le 6) = 0.6915$$

$$= P\left(Z \le \frac{6-\mu}{\sigma}\right) = 0.6915$$



$$\frac{6-\mu}{\sigma} = \frac{1}{2} \quad \Rightarrow \sigma = 12 - 2\mu$$

$$\boxed{2\mu + \sigma = 12} \text{---} (2)$$

$$\begin{aligned} &\cancel{2}\mu - \sigma = 8 \\ - \; &\cancel{2}\mu + \sigma = 12 \\ \hline &-2\sigma = -4 \end{aligned}$$

$$\boxed{\sigma = 2}$$

$$2\mu - 2 = 8$$

$$2\mu = 10$$

$$\boxed{\mu = 5}$$

$$\boxed{\begin{aligned} \sigma^2 &= 4 \\ \mu &= 5 \end{aligned}}$$

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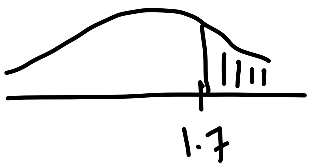
Example suppose that the grades in general exam are normally distributed with mean 68 and SD equal to 10, find:

a) the proportion of students that achieved more than 85

$$P(X > 85) = P\left(Z > \frac{85-68}{10}\right)$$




$$= P(Z > 1.7)$$

$$= 0.0446$$

B) the proportion of students that achieved between 60 and 90

$$= P(60 < X < 90)$$

$$= P(X < 90) - P(X < 60)$$

$$= P\left(Z < \frac{90-68}{10}\right) - P\left(Z < \frac{60-68}{10}\right)$$

$$= P(Z < 2.2) - P(Z < -0.8)$$

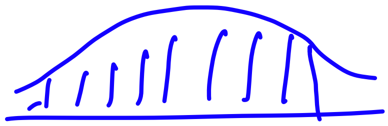
$$= 0.9861 - 0.2119$$

C) $95^{th}$ percentile $(P_{95})$

$$P_{95} \Rightarrow P(X \le P_{95}) = 0.95$$

$$= P\left(Z \le \frac{P_{95} - 68}{10}\right) = 0.95$$

$$Z = \frac{1.64 + 1.65}{2}$$



$$= 1.645$$

$$1.645 = \frac{P_{95} - 68}{10}$$

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Example: If heights of students are normally distributed with mean 170 Cm and SD 10 Cm. Find:

A) a student is selected at random what is the probability that he will be shorter than 170

$$P(X < 170) \Rightarrow P\left(Z < \frac{170 - 170}{10}\right)$$

$$= P(Z < 0)$$

$$= 0.5$$

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B) 10 students are selected at random what is the probability that exactly 4 of them are shorter than 170 ?

$$X \sim Bin(10, 0.5)$$

$$P(X = 4) = \binom{10}{4} * 0.5^{4} * 0.5^{6}$$

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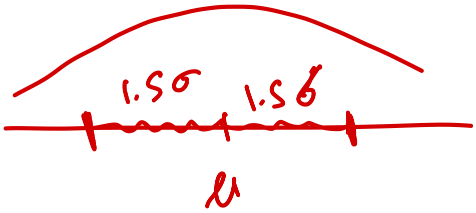
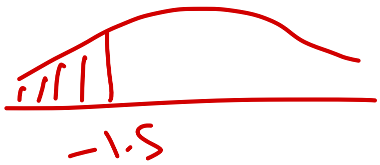
**Example** Suppose a child is considered to have normal lung growth if his/her standarized FVC is within 1.5 standard deviation of the mean. what is the proportion of children are within the normal range?

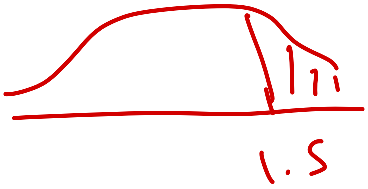



$$\Rightarrow P(-1.5 < Z < 1.5)$$

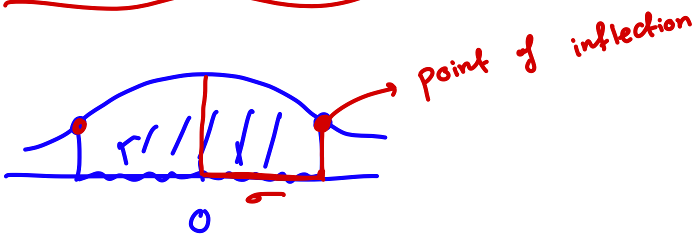
$$= P(Z < 1.5) - P(Z < -1.5)$$

$$= 0.9332 - 0.0668 = 0.8664$$




## NOTE




1. $P(-1 < Z < 1) = 0.6827$
2. $P(-2 < Z < 2) = 0.95$
3. $P(-3 < Z < 3) = 0.99$

NOTE the height of normal distribution curve is always $= \dfrac{1}{\sqrt{2\pi}\,\sigma}$

$\downarrow\uparrow h \propto \dfrac{1}{\downarrow\sigma\uparrow}$

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Example If $X \sim n(50, 4)$, find:

1) the mean $= 50$

2) the mode $= 50$

3) the median $= 50$

4) IQR $= Q_3 - Q_1 = 51.34 - 48.66$
$= 2.68$

5) variance and SD
4 2

$IQR \Rightarrow Q_3 : P_{75} \Rightarrow P(X \leq P_{75}) = 0.75$

$= P\left(Z \leq \dfrac{P_{75} - 50}{2}\right) = 0.75$

$$0.67 = \frac{P_{75} - 50}{2}$$

$$\boxed{P_{75} = 51.34}$$

$$Q_1 : P_{25} \Rightarrow P(X \leq P_{25}) = 0.25$$

$$\Rightarrow P\left(Z \leq \frac{P_{25} - 50}{2}\right) = 0.25$$

$$-0.67 = \frac{P_{25} - 50}{2}$$

$$P_{25} = 48.66$$