Chapter 4

Discrete probability distribution

المتغير العشوائي

* The Random variable

$\Rightarrow$ A function from the sample space to a set of real numbers (X, Y, Z)

Example when tossing a fair coin 2-times, the Random variable X is the number of heads obtaind, what is the RV?

$$\Omega : \left\{ \begin{matrix} \overset{2}{(H,H)}, & \overset{1}{(H,T)} \\ \underset{1}{(T,H)}, & \underset{0}{(T,T)} \end{matrix} \right\}$$

$$X : \underbrace{\{0, 1, 2\}}_{\text{support}}$$

NOTE

قيم RV بنسميها Support

Countable

"integers"

"أرقام صحيحة"

$\Rightarrow$ Discrete Random variable

0, 1, 8, 9

non Countable

"intervals"

"فترات/كسور عشرية"

$\Rightarrow$ Continous Random variable

3.8, 6.2

Example Find probability distribution for the previous question?

X	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

* probability distribution for discrete Random variable (Discrete probability distribution)

Example Determine whether each Random variable X is discrete or Continous:

① Let X be the number of fortune 500 Companies that lost money in previous year

② Let X Represent the volume of gasoline in a 21 - gallon tank

③ Let X Represent the speed of Rockets

④ Let X Represent the number of Calves born on a farm in one - year

⑤ Let X Represent the number of days of rain for the next 3 days

NOTE

integers دائماً Discrete Random variable قيم -

integes

non - integers

Continous Random variable قيم -

* Probability mass function (PMS)

① $0 \leq P(X=x) \leq 1$

② $\sum P(X=x) = 1$

Relative Frequency

$$\boxed{R.f = \frac{f}{\sum f} = P(X=x)}$$

x	0	1	2	3	4	5
f	10	5	1	3	8	3

Frequency distribution

⇓

x	0	1	2	3	4	5
$P(X=x)$	$\frac{10}{30}$	$\frac{5}{30}$	$\frac{1}{30}$	$\frac{3}{30}$	$\frac{8}{30}$	$\frac{3}{30}$

PMF

Example If $P(X=x) = K \cdot X$, $X = 1, 2, 3, 4$

and is a PMF, find:

$0 \leq P(X=x) \leq 1$

$\sum (P(X=x)) = 1$

x	1	2	3	4
$P(X=x)$	k	$2k$	$3k$	$4k$

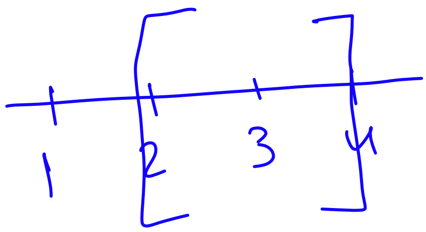
1. The value of k

$$k + 2k + 3k + 4k = 1$$

$$10k = 1 \qquad \boxed{k = 0.1}$$

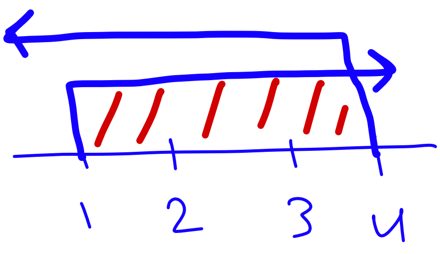
2. $P(X=3) = 3k = 3 * 0.1 = 0.3$

3. $P(1 < X \le 4)$



$$= P(X=2) + P(X=3) + P(X=4) = 0.9$$

4. $P(X>1 \mid X<4) = \dfrac{P(X>1 \cap X<4)}{P(X<4)}$



$$= \frac{P(X=2) + P(X=3)}{P(X=1) + P(X=2) + P(X=3)}$$

$$= \frac{0.5}{0.6} = \frac{5}{6}$$

5) $p(X = 3.4) = 0$

Example when throwing a fair dice 2 times, define the Random variable S to be the sum of 2 numbers obtained. Find the Probability distribution of S ?

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

S	2	3	4	5	6	$\boxed{7}$	8	9	10	11	12
$P(S=s)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\boxed{\frac{6}{36}}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Symmetric

μ. population mean

$\bar{X}$. sample mean

$\boxed{\mu_{\text{mean}} = 7}$

* The Expected value ($E(x)$)

$$E(x) = \mu = \text{mean} = \sum x \cdot p(X=x)$$

Example find the expected value for.

①

x	1	2	3	4
$p(X=x)$	0.4	0.3	0.2	0.1

$$E(x) = 1*0.4 + 2*0.3 + 3*0.2 + 4*0.1$$

$$= 2$$

②

x	1	2	3
$p(X=x)$	A	$\boxed{B}$	A

$\mu = E(x) = 2 \Rightarrow$ Symmetric

Example If the $E(X) = 2$, find a and b?

x	1	2	3	4
$p(X=x)$	a	0.3	b	0.1

ج/ $2 = 1*a + 2*0.3 + 3*b + 4*0.1$

$$2 = a + 0.6 + 3b + 0.4$$

$$2 = 1 + a + 3b$$

$$\boxed{a + 3b = 1} \quad \text{---- (1)}$$

$$a + 0.3 + b + 0.1 = 1$$

$$a + b + 0.4 = 1$$

$$\boxed{a + b = 0.6} \quad \text{---- (2)}$$

$$\begin{array}{r} \cancel{a} + b = 0.6 \\ - \quad \cancel{a} + 3b = 1 \\ \hline \end{array}$$

$$a + 0.2 = 0.6$$

$$\boxed{a = 0.4}$$

$$-2b = -0.4$$

$$\boxed{b = 0.2}$$

☆ Properties of $E(x)$:

① $E(a) = a$ $E(3) = 3$

الثابت   الثابت نفسه $E(2.5) = 2.5$

② $E(ax) = a * E(x) \Rightarrow E(2x) = 2 * E(x)$

③ $E(x \pm y) = E(x) \pm E(y)$

Example

x	1	2	3	4
$P(X=x)$	0.4	0.3	0.2	0.1

find:

① $E(x) = 1 * 0.4 + 2 * 0.3 + 3 * 0.2 + 4 * 0.1$

$= 2$

② $E(x^2) = 1^2 * 0.4 + 2^2 * 0.3 + 3^2 * 0.2$

$$+ 4^2 * 0.1$$

$$= 5$$

3. $E\left(\frac{1}{x}\right) = \frac{1}{1} * 0.4 + \frac{1}{2} * 0.3 + \frac{1}{3} * 0.2 + \frac{1}{4} * 0.1$

$$= \_\_\_\_$$

4. $E\left(e^x\right) = e^1 * 0.4 + e^2 * 0.3 + e^3 * 0.2 + e^4 * 0.1 = \_\_\_\_$

العدد النيبري ≈ 2.7

Example If the $\mu = E(x) = 10$, find:

1. $E(\mu) = E(10) = 10$
2. $E(E(x)) = E(10) = 10$

③ $E(2X-3) = E(2X) - E(3)$

$= 2 * E(X) - 3$

$= 2 * 10 - 3 = 17$

④ $E\left(1 - \frac{X}{2}\right) = E(1) - E\left(\frac{X}{2}\right)$

$= 1 - E\left(\frac{1}{2} * X\right)$

$= 1 - \frac{1}{2} * E(X) = -4$

Example If the probability mass function for the number of episodes of otitis media in the first 2 years of life are shown, find the Expected number of visits?

r	0	1	2	3	4	5	6
$P(R=r)$	0.129	0.264	0.271	0.185	0.095	0.039	0.017

$$E(x) = 0*0.129 + 1*0.264 + 2*0.271 + 3*0.185 + 4*0.095 + 5*0.039 + 6*0.017 \simeq 2.0..-$$

خاتم

Example Determine which one of the following tables Represent PMF:

A)

r	0	1	2	3
$P(X=r)$	0.18	0.34	0.36	0.13

B)

r	0	1	2	3
$P(X=r)$	0.10	0.42	0.03	0.44

C)

r	0	1	2	3
$P(X=r)$	0.12	0.13	0.61	0.14

D)

r	0	1	2	3
$P(X=r)$	0.26	0.44	-0.15	0.45

A)

r	0	1	2	3	u
$P(X=r)$	0.15	0.25	0.10	0.25	0.30

B)

r	0	1	2	3
$P(X=r)$	0.15	0.20	0.30	0.10

C)

r	0	1	2	3	u
$P(X=r)$	0.15	-0.20	0.30	0.20	0.15

D)

r	-1	0	1	2	3	u
$P(X=r)$	0.15	0.30	0.20	0.15	0.10	0.10

★ The variance (σ^2)

$$\sigma^2 = var(X) = E(X-\mu)^2$$

$$\sigma^2 = E(X^2) - (E(X))^2$$

$$\boxed{\sigma^2 = E(X^2) - \mu^2}$$

Example

x	1	2	3	4
$P(X=x)$	0.4	0.3	0.2	0.1

Find the variance and STD?

$$\sigma^2 = E(X^2) - \mu^2$$

$$\Rightarrow E(X^2) = 1^2 * 0.4 + 2^2 * 0.3 + 3^2 * 0.2 + 4^2 * 0.1$$
$$= 5$$

$$\Rightarrow E(X) = \mu = 1 * 0.4 + 2 * 0.3 + 3 * 0.2 + 4 * 0.1$$
$$= 2$$

$$\sigma^2 = 5 - (2)^2 = 1$$

$$\sigma = \sqrt{1} = 1$$

Properities of variance:

① $Var(a) = 0$ $\quad var(9) = 0$, $Var(3) = 0$

② $var(ax) = a^2 * var(x)$

③ $var(ax \pm b) = Var(ax) \pm \cancel{var(b)}^{\,0}$

$= a^2 * var(x)$

④ $var(x) = E(x^2) - (E(x))^2$

$E(x^2) = Var(x) + (E(x))^2$

$$\boxed{E(x^2) = \sigma^2 + \mu^2}$$

Example If $\mu = 10$, $\sigma^2 = 3$ find:

i) $E(var(x)) = E(3) = 3$

ii) $Var(E(x)) = Var(10) = 0$

iii) $Var(var(x)) = var(3) = 0$

حل

NOTE

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

iv) $E(x - \overset{\mu}{10})^2 = E(x^2 - 20x + 100)$

$$E(x^2) - 20 * E(x) + E(100)$$

$$\sigma^2 + \mu^2 - 20 * 10 + 100$$

$$3 + 100 - 200 + 100 = 3$$

حل آخر

$$Var(x) = E(x - \mu)^2$$

$$= E(x - \mu)^2 = var(x) = 3$$

v) $E(x^2) = \sigma^2 + \mu^2 = 3 + 10^2 = 103$

vi) $E(x - 2)^2 = E(x^2 - 4x + 4)$

$$= E(x^2) - 4 * E(x) + E(4)$$

$$= 103 - 40 + 4$$

$$= 67$$

PMF

r	0	1	2	3	4	5	6
$P(R=r)$	0.129	0.264	0.271	0.185	0.095	0.039	0.017

NOTE: تقريباً ٪95 من الاحتمالات

في PMF تقع بين 2σ

Ex $E(x) = 2.038$

$\sigma = 1.402$

find a & b such that Approxemitaly 95% of data lie within it?

95%

(2.5 | 2.5)

a 2.038 b

$$a = 2.038 - 2 \ast 1.402 = \_\_$$

$$b = 2.038 + 2 \ast 1.402 = \_\_$$

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* Binomial distribution

التوزيع الثنائي

\# If we have n-independent (1) مستقلين trials $n \geq 2$ (2) and the (3) outcomes نتائج in EACH trial are Success or fail only. Let X be the number of success, then we say that X follows binomial distribution and is denoted by $X \sim Bin(n, p)$, where:

n: number of trials

p: probability of success

ال Binomial له ٣ شروط:

(1) $n \geq 2$

② 2 outcomes $\rightarrow$ success / fail

نتيجتان فقط

③ independent

$$X \underset{\text{follows}}{\sim} Bin(n, P)$$

يتبع

n : number of trials

p : probability of success in each trial

q : probability of fail in each trial

خواص

① $P + q = 1$

② $P(X = k) = \binom{n}{k} * P^{k} * q^{n-k}$

$\binom{n}{k}$: Combination

③ $\mu = E(X) = n \cdot P$

④ $\text{var}(X) = \sigma^2 = n \cdot P \cdot q$

(5) $STD(X) = \sigma = \sqrt{n \cdot p \cdot q}$

∴ ∴

Example when tossing a fair coin 10 times, find:

1. independent
2. $n \geq 2$
3. outcomes $\begin{cases} H \\ T \end{cases}$

$X \sim Bin(10, 0.5)$

A) the probability of getting:

i) exactly 8 heads

$$P(X=8) = \binom{10}{8} * 0.5^{8} * 0.5^{(10-8)}$$

$$= \underline{\underline{0.04}}$$

ii) at least 9 H $= P(X \geq 9)$

$$= P(X=9) + P(X=10)$$

$$= \binom{10}{9} * 0.5^{9} * 0.5^{(10-9)} + (- - - -$$

at least :	على الأقل	$P(X \geq k)$
at most :	على الأكثر	$P(X \leq k)$

iii) at least 2 H

$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - P(X \leq 1)$$
$$= 1 - \left(P(X=0) + P(X=1)\right)$$

$\vdots$

iv) at most 1 H

$$P(X \leq 1) = \left(P(X=0) + P(X=1)\right)$$

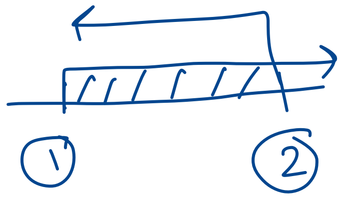
v) at most 8 H

$$P(X \leq 8) = 1 - P(X > 8)$$
$$= 1 - P(X \geq 9)$$
$$= 1 - \left(P(X=9) + P(X=10)\right)$$

vi) at most 2 H given that at least 1 H.

$$\Rightarrow P\left(\text{at most } 2H \;\middle|\; \text{at least } 1H\right) = \frac{P\left(\text{at most } 2H \cap \text{at least } 1H\right)}{P\left(\text{at least } 1H\right)}$$

$$= \frac{P(X \leq 2 \cap X \geq 1)}{P(X \geq 1)}$$



$$= \frac{P(X=1) + P(X=2)}{(1 - P(X<1))}$$

$$= \frac{P(X=1) + P(X=2)}{(1 - P(X=0))} = - - -$$

B) The Expected number of heads and the variance and the standard deviation

$$\Rightarrow E(X) = \mu = n \cdot p$$

$$= 10 * 0.5 = 5$$

$$\Rightarrow \sigma^2 = n \cdot p \cdot q = 10 * \frac{1}{2} * \frac{1}{2} = 2.5$$

$$\Rightarrow \sigma = \sqrt{n \cdot p \cdot q} = \sqrt{2.5} = \text{—}$$

Example If $X \sim Bin(3, p)$ and $p(X \geq 1) = \frac{19}{27}$

find $var(X)$?

$$var(X) = n \cdot p \cdot q$$

$$= \cancel{3} * \frac{1}{\cancel{3}} * \frac{2}{3} = \boxed{\frac{2}{3}}$$

$$p(X \geq 1) = \frac{19}{27}$$

$$1 - p(X < 1) = \frac{19}{27}$$

$$1 - \frac{19}{27} = p(X = 0)$$

$$\frac{8}{27} = \cancel{\binom{3}{0}} * \cancel{p^0} * q^{(3-0)}$$

$$\frac{8}{27} = q^3 \Rightarrow \boxed{q = \frac{2}{3}} \quad \boxed{P = \frac{1}{3}}$$

Example If $X \sim Bin(n, P)$ and $\mu = 2$, $\sigma^2 = 1.6$ find n, P?

$$2 = n.P \longrightarrow \cancel{2} = n * \frac{\cancel{2}}{10}$$

$$\boxed{n = 10}$$

$$1.6 = \underbrace{n.P}.q$$

$$1.6 = 2 * q$$

$$\boxed{q = 0.8} \quad \boxed{P = 0.2}$$

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* Using the Binomial tables

Example If $X \sim Bin(10, 0.4)$, find.

i) $P(X \le 6) = 0.945$

ii) $P(X > 6) = 1 - P(X \le 6)$
$= 1 - 0.945 = \_\_\_$

iii) $P(X<7) = P(X \le 6) = 0.945$

iv) $P(X \geq 7) = 1 - P(X<7)$

$= 1 - P(X \le 6)$

$= 1 - 0.945 = \text{——}$

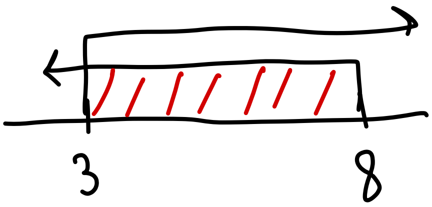
v) $P(3 \le X \le 8) =$

vi) $P(3 < X \le 8) = P(4 \le X \le 8)$

vii) $P(3 < X < 8) = P(4 \le X \le 7)$

viii) $P(3 \le X < 8) = P(3 \le X \le 7)$

ix) $P(X \geq 3 \mid X \le 8) = \dfrac{P(X \geq 3 \cap X \le 8)}{P(X \le 8)}$



$= \dfrac{P(3 \le X \le 8)}{P(X \le 8)}$

Example A family has 5 children, what is the probability that 3 children are females.

$$X \sim Bin(5, 0.5)$$

$$P(X=3) = \binom{5}{3} * 0.5^{3} * 0.5^{(5-3)} = 0.3125$$

Example In a multiple choice exam of 10 quest. each question had 5 answers only one of them is correct. Ahmad is answering the exam by guessing what is the probability that ahmad will answer:

1) 5 questions correctly

$$P(X=5) = \binom{10}{5} * 0.2^{5} * 0.8^{(10-5)} = 0.027$$

2) at most 5 questions

$$P(X \leq 5) = 0.994$$

Example what is the probability of obtaining 2 boys out of 5 children if the probability of a boy is 0.51 at each birth and the genders are considered independent Random variables?

$$X \sim Bin(5, 0.51)$$

$$P(X=2) = \binom{5}{2} * 0.51^{2} * 0.49^{(5-2)}$$

$$= 0.306$$

Example Evaluate the probability of 2 lymphocytes out of 10 white blood cells if the probability of any one cell being a lymphocyte is 0.2.

$$X \sim Bin(10, 0.2)$$

$$P(X=2) = \binom{10}{2} * 0.2^{2} * 0.8^{10-2} = 0.3020$$